MINIMAL PATHS ON SOME SIMPLE SURFACES WITH SINGULARITIES

JOEL B. MOHLER AND RON UMBLE

ABSTRACT. Given two points on a soup can or conical cup with lid, we find and classify all paths of minimal length connecting them. When the number of minimal paths is finite, there are at most four on a can and three on a cup. At worst, minimal paths are piece-wise smooth with three components, each of which is a classical geodesic. Minimal paths are geodesics in the sense of Banchoff [1].

1. INTRODUCTION

This paper considers the following problem: *Given two points A and B on a soup can or conical cup (with lid) S, find all paths of minimal length connecting them.* The fact that such a path exists follows from Ascoli's Theorem (e.g. [4]): Let h be the height and d the diameter of S. Let $M = 2(h+d)$ and consider the set G of all piecewise smooth constant speed paths $\gamma : [0,1] \to S$ such that $\gamma(0) = A$, $\gamma(1) = B$, length $(\gamma) \leq M$ and $\|\gamma'\| \leq M$; then the speed is uniformly bounded and G is equicontinuous. Since equicontinuity is preserved in the closure, $\overline{G}$ is compact and contains a minimizer.

In this paper we find and classify all minimal paths from A to B. When the number of minimal paths is finite, there are at most four on a can and three on a cup. At worst, minimal paths are piecewise smooth with three components, each of which is a classical geodesic. We adapt Banchoff's notion of a geodesic [1] and prove a Hopf-Rinow-like theorem assuring us that minimal paths are always geodesics (e.g. [5]). While soup cans and conical cups are piecewise-smooth closed compact surfaces with singularities along their rims, they have some interesting distinguishing properties. For example, K is the cone point of a conical cup if and only if K is an endpoint of every minimal path containing it. No point on a soup can has this property.

This research grew out of an undergraduate seminar directed by the second author in the spring of 2001 during which participants Heather Armstrong (Cornell University), Robert Painter (College of William and Mary), Ellen Panofsky (Lehigh University) and the first author solved the problem for soup cans. We acknowledge the substantive contribution of each participant to this work with thanks. Subsequently, new independent proofs of the seminar results were obtained by the first author and appeared in his undergraduate honors thesis [3] supervised by the second author. This paper generalizes these results to conical cups in the sense that most
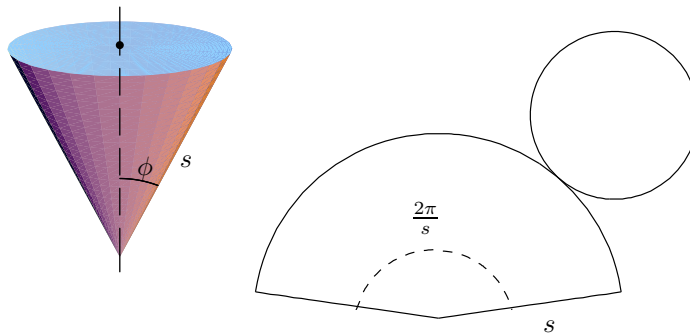
FIGURE 1. A flat model of a conical cup.

statements about geodesics on a conical cup have analogues on the soup can. These analogies are realized in the limit as cone height goes to infinity. This research was enabled by the kind and generous support of Professor Frank Morgan, Williams College, who posed the soup can problem to our seminar group and offered many helpful suggestions that improved this paper; we thank him as well.

We think of a soup can or conical cup as either a surface of revolution or an identification space formed by isometrically bending and gluing the appropriate plane regions together along their boundaries. The identification space point-of-view is particularly advantageous because it provides an isometric decomposition of our surface S into a family $\mathfrak{F}$ of compact simply-connected plane regions with piece-wise smooth boundaries; such decompositions come equipped with an arc length-preserving quotient map $q : \mathfrak{F} \to S$. We refer to such a pair $(\mathfrak{F}, q)$ as a *flat model* of S. In this paper, the family $\mathfrak{F}$ in a flat model of a conical cup with cone angle ϕ and slant height $s = \csc \phi$ consists of a closed unit disk U (the *lid*) tangent to a circular sector C of radius s and subtended angle $2\pi/s$ (the *side*); the circle $T = \partial U$ is called the *rim* (see Figure 1). The family $\mathfrak{F}$ in a flat model of a soup can with height h consists of a $2\pi \times h$ rectangular region R (the *side*) and two closed unit disks U_1 (the *lid*) and U_2 (the *base*) tangent to R along its opposite edges of length 2π; the circles $T_i = \partial U_i$ are called the *rims*. When there is no confusion, we use the same symbol X to denote a subset of S and its corresponding subset in the family $\mathfrak{F}$.

Let S be a surface and let $I \subseteq \mathbb{R}$ be an interval. A *path* on S is an element $\gamma \in C(I; S) = \{\gamma : I \to S \mid \gamma \text{ continuous}\}$; in particular, if $I = [a, b]$, $A = \gamma(a)$ and $B = \gamma(b)$, then γ is a *path from A to B*. A path γ from A to B is *minimal* if length $(\gamma) \leq$ length (β) for all $\beta \in C(I; S)$. Geodesics on S appear as a union of straight line segments in some family $\mathfrak{F}$ of a flat model. In [1], Banchoff uses flat models to study closed geodesics on "bicylinders," but he does not consider their distance minimizing properties. We adapt his notion of a geodesic:

Definition 1. *A path $\gamma \in C([a, b]; S)$ is a* geodesic *if for each $t \in (a, b)$ there is a flat model $(\mathfrak{F}_t, q_t)$ of S such that for every $\epsilon > 0$ there is a $\delta > 0$ and a line segment $\ell \subset \mathfrak{F}_t$ of length δ and midpoint $q_t^{-1}(\gamma(t))$ such that $\ell \subset q_t^{-1}[\gamma(t + \epsilon, t - \epsilon)]$.*

We mention that Cotton et al. [2] use flat models to study the classical isoperimetric problem on soup cans. The graphics in this paper were generated by Mathematica (www.wolfram.com) and refined using XFig (www.xfig.org).

2. Computing minimal paths

In this section we introduce constructive methods for finding minimal paths and computing their length; this reduces the problem to numerical computation. Unless explicitly stated otherwise, S denotes a soup can or a conical cup; A and B denote distinct points on S. We say that A and B are *separated* by $T = \partial U$ if $A \in S - U$ and $B \in U - T$ (or vice versa). We denote the Euclidean distance between points X and Y by XY; if E is a set, $\#E$ denotes its cardinality.

We begin with a proposition that allows us to restrict our search for minimal paths to paths that intersect a rim at most twice:

Proposition 1. *Let T be a rim of S. A path γ on S from A to B can be shortened if either*

    a. $\#(\gamma \cap T) \geq 3$ *or*
    b. *A and B are separated by T and $\#(\gamma \cap T) \geq 2$.*

Proof. (a) Consider a path γ from A to B. Suppose there are distinct points $X, Y, Z \in \gamma \cap T$ ordered by increasing parameter. Let γ_1 and γ_2 be the respective pieces of γ from X to Y and from Y to Z. Then $\triangle XYZ$ is non-degenerate and $\text{length}(\gamma_1 \cup \gamma_2) \geq XY + YZ > XZ$, by the triangle inequality. Thus $[\gamma - (\gamma_1 \cup \gamma_2)] \cup \overline{XZ}$ is shorter than γ.

(b) Suppose A and B are separated by T. With-out loss of generality, assume that $B \in U - T$. By assumption, there exist distinct points $X, Y \in \gamma \cap T$. Let γ_1 be the piece of γ from X to Y and let γ_2 be the piece of γ from Y to B. Then $\triangle XYB$ is non-degenerate or B lies in the interior of $\overline{XY}$. In either case, $\text{length}(\gamma_1 \cup \gamma_2) = XY + YB > XB$ by the triangle inequality and $(\gamma - (\gamma_1 \cup \gamma_2)) \cup \overline{XB}$ is shorter than γ. $\qquad\square$

Corollary 1. *If A and B are separated by T, every minimal path from A to B intersects T exactly once.*

Proof. If A and B are separated by T, then $\#(\gamma \cap T) \geq 1$ by continuity of γ. $\quad\square$

It is essential to know where to look for minimal paths. *Axial points* of S lie on its axis of revolution. If P is a non-axial point of S, let Γ_P denote the closed half-plane containing P and bounded by the axis.

Proposition 2. *If $A, B \in \Gamma_Q$, then $\Gamma_Q \cap S$ contains a minimal path γ from A to B. Furthermore, if A or B is non-axial, then γ is the unique minimal path from A to B.*

Proof. We prove the result for conical cups; the proof for soup cans is similar and left to the reader. If A and B lie on the lid U, then $\overline{AB}$ is contained in the radius $\Gamma_Q \cap U$ and is (uniquely) minimal. If A and B lie on the side C, then $\overline{AB}$ is contained in the ruling $\Gamma_Q \cap C$ and is (uniquely) minimal. If A and B are separated by T with $B \in U - T$, let $X = \Gamma_Q \cap T$ and let $\gamma = \overline{AX} \cup \overline{XB}$; we claim that γ is minimal. Let α be any path from A to B; then α intersects T at least once by Proposition 1. Let $(\mathfrak{F}, q)$ be the flat model in which the side and lid are tangent at X and let $Y \in \alpha \cap T$. Then $\overline{AY}$ and $\overline{YB}$ correspond to classical distance minimizing

geodesics on C and U, respectively. Let α_1 and α_2 be the respective pieces of α from A to Y and from Y to B, then length $(\alpha_1) \geq AY$ and length $(\alpha_2) \geq YB$. But $AY \geq AX$ since $\overline{AX}$ is normal to the arc of sector C; and $YB \geq XB$ since $\overline{XB}$ is normal to T. Therefore length $(\alpha) \geq AY + YB \geq AB =$ length (γ) and γ is minimal. Furthermore, if A is non-axial and $Y \neq X$, segment $\overline{AY}$ is not normal to the arc of sector C, in which case $AY > AX$ and γ is unique. $\square$

Let A and B be non-axial points of S and let θ be the angle subtended by Γ_A and Γ_B. If $0 < \theta < \pi$, let $\Omega_{A,B}$ denote the closed region of space bounded by $\Gamma_A \cup \Gamma_B$ and subtending an angle θ. If $\theta = 0$, define $\Omega_{A,B} = \Gamma_A = \Gamma_B$. If $\theta = \pi$, define $\Omega_{A,B}$ to be either (arbitrarily chosen) closed half-space bounded by the plane $\Gamma_A \cup \Gamma_B$. If exactly one of A or B is non-axial, call it Q and define $\Omega_{A,B} = \Gamma_Q$. If both A and B are axial, let Q be an arbitrarily chosen non-axial point of S and define $\Omega_{A,B} = \Gamma_Q$. Below we observe that a minimal path from A to B lies in some set $\Omega_{A,B}$. This fact is well-known for classical distance minimizing geodesics on the lid or side of a lid-less cone and is established for $A, B \in \Gamma_Q$ by Proposition 2. To this end, we first establish a Hopf-Rinow-like connection between minimal paths and geodesics (see Theorem 1 below) by appealing to some well-known properties of roulettes.

Definition 2. *Let $B \in \mathbb{R}^2$ be fixed with respect to a closed convex curve C_1; let C_2 be any plane curve. The* roulette *generated by B is the curve traced out by B as C_1 rolls without slipping along C_2.*

In our considerations below, the point B in Definition 2 lies on a lid U in some flat model $(\mathfrak{F}_0, q_0)$, the curve C_1 is the rim $T = \partial U$ and C_2 is an edge of the side. When S is a soup can, the roulette generated by B is a *cycloid*; when S is a conical cup, the roulette generated by B is an *epicycloid* (see Figure 2).
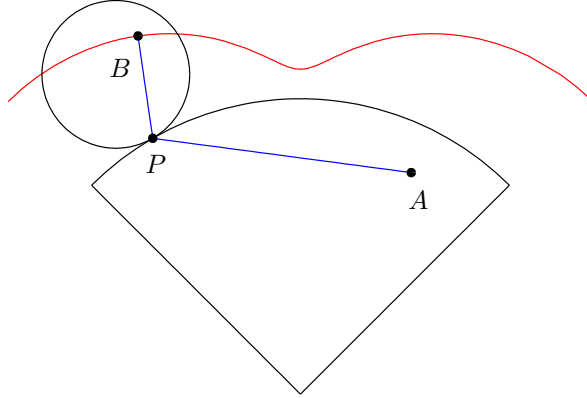

FIGURE 2. A non-minimal path crossing the rim.

Lemma 1. *If A and B are separated by a rim, every minimal path from A to B is a geodesic.*

Proof. We prove the result for conical cups; the result for soup cans is obtained in the limit as cone height goes to infinity. Given a non-geodesic path γ from A

to B crossing the rim T at point P, we construct an alternate path γ' such that $\operatorname{length}(\gamma') < \operatorname{length}(\gamma)$. Let γ_1 the piece of γ from A to P and let $\gamma_2 = \gamma - \gamma_1$. Since P is shared by the side and lid, there are classical distance minimizing geodesics from A to P and from P to B. So assume that paths γ_i are such geodesics, in which case γ intersects T exactly once at P. Consider the flat model $(\mathfrak{F}, q)$ in which the lid U is tangent to the sector C at P. Since γ is not a geodesic, $q^{-1}(\gamma) = \overline{AP} \cup \overline{PB}$ is not straight (see Figure 2). Construct a shorter path γ' from A to B as follows: Let R denote the arc in the boundary of sector C and $T = \partial U$. Let $P' = \overline{AB} \cap T$ and $P'' = \overline{AB} \cap R$; let s_1 be the length of the arc along T from P to P' and inside $\triangle APB$; and let s_2 be the length of the arc along R from P to P'' (see Figure 3). If $s_1 \le s_2$, let α be the epicycloid generated by P' as disk U rolls along R towards P''. Let Q be the first cusp that appears. We claim that $P'P'' > QP''$. Let E be the intersection of $\overline{P'P''}$ and the line normal to R at Q. Construct the chord $\overline{QP''}$; note that $P'P'' > EP''$ since P' and P'' are on opposite sides of $\overline{QE}$. Furthermore, $\theta = m\angle EQP'' > \pi/2$ so that $\cos\theta < 0$. Then by the Law of Cosines,

$$(EP'')^2 = (QP'')^2 + (QE)^2 + |2(QP'')(QE)\cos\theta| > (QP'')^2$$

so that $EP'' > QP''$, which verifies the claim. Thus $AB > AP'' + QP'' + P'B$ and $\gamma' = \overline{AP''} \cup \overline{P''Q} \cup \overline{P'B}$ is shorter than γ. On the other hand, if $s_1 > s_2$ consider the epicycloid generated by P'' as sector C rolls along T towards P' and construct the path $\gamma' = \overline{AP''} \cup \overline{P'Q} \cup \overline{P'B}$. □

FIGURE 3. Shortening a non-geodesic path from A and B.

Theorem 1. *Every minimal path from A to B is a geodesic.*

Proof. Let γ be a minimal path from A to B. If $\gamma \subset S - U$ or $\gamma \subset U$, the result is classical. The case with A and B separated by T is given by Lemma 1. Suppose that γ intersects T exactly twice at P_1 and P_2. If $A, B \in T$, then γ is a geodesic in U. So assume that A lies off T, then $A \in S - U$, for otherwise $\triangle AP_1P_2 \subset U$ is non-degenerate and γ is not minimal. Choose a point $Q \in \gamma \cap (U - T)$ and let γ_1, γ_2 and γ_3 be the respective pieces of γ from A to Q, from Q to B and from P_1 to P_2. Then γ_1 is a geodesic by Lemma 1 and γ_3 is a geodesic in U. Therefore $\gamma_1 \cap \gamma_3 = \overline{P_1Q}$ is a geodesic. If $B \in T$, then $\gamma_2 \subset \gamma_3$ and we're done. If B is off T, then $B \in S - U$ and γ_2 is a geodesic by Lemma 1. Therefore $\gamma_2 \cap \gamma_3 = \overline{QP_2}$ is a geodesic and the conclusion follows. $\square$

Proposition 3. *Let P be a non-axial point of S. If a minimal path γ from A to B intersects Γ_P, either $\gamma \cap \Gamma_P = \gamma$ or $\gamma \cap \Gamma_P$ is a single point.*

Proof. Suppose distinct points $Q, Q' \in \gamma \cap \Gamma_P$; let $(\mathfrak{F}, q)$ be the flat model containing $\Gamma_P \cap S$. Since γ is minimal, $\overline{QQ'} \subset \gamma \cap \Gamma_P$ by Proposition 2. So consider the connected component α of $\gamma \cap \Gamma_P \subset \mathfrak{F}$ containing $\overline{QQ'}$; let $A = \gamma(a)$, $B = \gamma(b)$. Since γ is a geodesic by Theorem 1, no point $E = \gamma(t)$, $t \in (a, b)$ is an endpoint of α, for if it were, every neighborhood of E would violate the requirements of Definition 1. Therefore A and B are the endpoints of α and $\gamma = \alpha$. $\square$

Theorem 2. *Every minimal path from A to B lies in some set $\Omega_{A,B}$.*

Proof. The case of $A, B \in \Gamma_Q$ for some Q was established in Proposition 2. So assume A and B are non-axial points of S and let θ be the angle between Γ_A and Γ_B. If $0 < \theta < \pi$, suppose $\gamma \cap \Omega_{A,B}^c \neq \varnothing$. Then by Proposition 3, we may assume that $\gamma \subset \{A, B\} \cup \Omega_{A,B}^c$. Since $0 < \theta < \pi$, there is a plane Π containing the axis and bounding an open half-space H with $A, B \in H$. Let $\overline{\gamma}$ be the reflection of $\gamma \cap H^c$ in Π; then $\beta = \overline{\gamma} \cup (\gamma \cap H)$ is a new path passing through Γ_A (and Γ_B) at least twice, i.e., at $A = \beta(a)$ and $P = \beta(u)$ for some $u \neq a$ (see Figure 4). If $A = P$, we can shorten β by redefining $\beta(t) = A$ for all $t \in [a, u]$; otherwise we can shorten β by applying Proposition 2, contradicting minimality in either case. Finally, suppose that $\theta = \pi$ and γ contains points on both sides of plane $\Pi = \Gamma_A \cup \Gamma_B$. If $A = \gamma(a)$ and $\gamma(t) \in \Gamma_A$ for some $t \neq a$, then $\gamma \subset \Gamma_A$ by Proposition 3, which is a contradiction. Therefore $\gamma \cap \Gamma_A = A$. Likewise, $\gamma \cap \Gamma_B = B$ and the conclusion follows by the continuity of γ. $\square$

Note that when $\theta = \pi$ and a half-space $\Omega_{A,B}$ has been chosen, $\Omega_{A,B}$ and the closure of its complement each contain a minimal path from A to B. With the characterization of minimal length paths developed to this point, we now exhibit numerical methods to find such paths in Propositions 4-7. Denote the cylindrical coordinates of point P by $P[r, \theta_P, z]$.

Proposition 4. *Let A and B be respective points on the side and lid of S. If A and B are axial points, there is a minimal path from A to B in Γ_Q for each non-axial Q. If exactly one of A or B is non-axial, call it Q, there is a unique minimal path from A to B in Γ_Q. If A and B are non-axial and separated by rim T, position S so that A has cylindrical angle $\theta_A = 0$, let h be the height of S, let b be the distance from B to the axis, let $\theta \in [0, \pi]$ be the angle between Γ_A and Γ_B and let $t \in [0, \theta]$; let γ_1 be the classical distance minimizing geodesic from A to $P[1, t, h]$ and let $\gamma_2 = \overline{PB}$.*
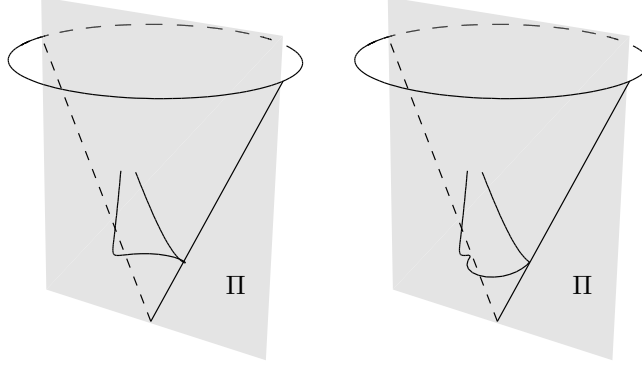
FIGURE 4. A geodesic reflected in vertical plane Π.

a. *If S is a conical cup with slant height s and a is the distance from A to the cone point, then*

$$\ell(t) = \text{length} \left(\gamma_1 \cup \gamma_2 \right)$$
$$= \sqrt{a^2 - 2as \cos(t/s) + s^2} + \sqrt{b^2 - 2b \cos(\theta - t) + 1}.$$

b. *If S is a soup can and a is the distance from A to the rim, then*

$$\ell(t) = \text{length} \left(\gamma_1 \cup \gamma_2 \right)$$
$$= \sqrt{a^2 + t^2} + \sqrt{b^2 - 2b \cos(\theta - t) + 1}.$$

In either case, ℓ is minimized at some $t_1 \in [0, \theta]$ and a path $\gamma_1 \cup \gamma_2$ of length $\ell(t_1)$ is a minimizer.

Proof. The axial cases were established by Proposition 2; so assume A and B are non-axial. The formulation for length $\ell(t)$ follows from the Law of Cosines; the restriction of parameter $t \in [0, \theta]$ follows from Theorem 2. The conclusion follows from continuity of ℓ on $[0, \theta]$. $\qquad\square$

Our next result is specific to conical cups:

Proposition 5. *Suppose A and B are non-axial points on the side of a conical cup S with height h and slant height s, positioned so that A has cylindrical angle $\theta_A = 0$. Let $\theta \in [0, \pi]$ be the angle between Γ_A and Γ_B and let $t, u \in [0, \theta]$; let γ_1 be the classical minimal geodesic from A to the point $P_1 [1, t, h] \in T$, let γ_2 be the classical minimal geodesic from $P_2 [1, u, h] \in T$ to B and let $\gamma_3 = \overline{P_1 P_2}$ on the lid. If the respective distances from A and B to the cone point are a and b, then*

$$\ell(t, u) = \text{length} \left(\gamma_1 \cup \gamma_2 \cup \gamma_3 \right)$$
$$= \sqrt{a^2 - 2as \cos(t/s) + s^2} + \sqrt{b^2 - 2bs \cos(u/s) + s^2}$$
$$+ \sqrt{2 - 2 \cos(\theta - t - u)}.$$

Candidates for minimizer appear as:

a. *a classical minimal geodesic α on the side, or*
b. *a path $\beta = \gamma_1 \cup \gamma_2 \cup \gamma_3$ across the lid of minimal length $\ell(t_1, u_1)$. A minimal path from A to B is minimal in $\{\alpha, \beta\}$.*

Proof. The result follows by continuity of ℓ on the compact set $[0,\theta] \times [0,\theta]$. $\square$

Note that one or both of A and B in Proposition 5 may lie on the rim. This result generalizes to soup cans by applying the same analysis at both ends:

Proposition 6. *Suppose A and B are points on the side of a soup can S with height h, positioned so that A has cylindrical angle $\theta_A = 0$. Let $\theta \in [0, \pi]$ be the angle between Γ_A and Γ_B and let $t, u \in [0, \theta]$; let γ_1 be the classical minimal geodesic from A to the point $P_1 [1, t, h] \in T$, let γ_2 be the classical minimal geodesic from $P_2 [1, u, h] \in T_1$ to B and let $\gamma_3 = \overline{P_1 P_2}$ on the lid. If a and b are the respective distances from A and B to rim T_1. Then*

$$\ell_1 (t, u) = \text{length} (\gamma_1 \cup \gamma_2 \cup \gamma_3)$$
$$= \sqrt{a^2 + t^2} + \sqrt{b^2 + u^2} + \sqrt{2 - 2\cos(\theta - t - u)}.$$

Furthermore, let γ_4 be the classical minimal geodesic from A to $Q_1 [1, t, 0] \in T_2$, let γ_5 be the classical minimal geodesic from $Q_2 [1, u, 0] \in T_2$ to B and let $\gamma_6 = \overline{Q_1 Q_2}$ on the base. Then

$$\ell_2 (t, u) = \text{length} (\gamma_4 \cup \gamma_5 \cup \gamma_6)$$
$$= \sqrt{(h-a)^2 + t^2} + \sqrt{(h-b)^2 + u^2}$$
$$+ \sqrt{2 - 2\cos(\theta - t - u)}.$$

Candidates for minimizer appear as:
  a. *a classical minimal geodesic α on the side,*
  b. *a path $\beta_1 = \gamma_1 \cup \gamma_2 \cup \gamma_3$ across the lid of minimal length $\ell_1 (t_1, u_1)$, or*
  c. *a path $\beta_2 = \gamma_4 \cup \gamma_5 \cup \gamma_6$ across the base of minimal length $\ell_2 (t_2, u_2)$.*
     A minimal path from A to B is minimal in $\{\alpha, \beta_1, \beta_2\}$.

Proof. The result follows by continuity of ℓ_i on the compact set $[0, \theta] \times [0, \theta]$. $\square$

Again, one or both of A and B in Proposition 6 may lie on a rim. Our next result is specific to soup cans.

Proposition 7. *Let A and B be non-axial points on the lid and base of a soup can S whose respective distances from the axis are a and b. Position S so that A has cylindrical angle $\theta_A = 0$. Let h be the of height of S, let θ be the angle between Γ_A and Γ_B, let $t, u \in [0, \theta]$, and consider the point $P [1, t, h] \in T_1$ and $Q [1, u, 0] \in T_2$. Let $\gamma_1 = \overline{AP}$ on the lid, let $\gamma_2 = \overline{QB}$ on the base and let γ_3 be a classical minimal geodesic on the side from P to Q. Then*

$$\ell (t, u) = \text{length} (\gamma_1 \cup \gamma_2 \cup \gamma_3)$$
$$= \sqrt{a^2 - 2a \cos(\theta - t) + 1} + \sqrt{b^2 - 2b \cos(\theta - u) + 1}$$
$$+ \sqrt{h^2 + (\theta - t - u)^2}.$$

Then ℓ is minimized at some $(t_1, u_1) \in [0, \theta] \times [0, \theta]$ and a path $\gamma_1 \cup \gamma_2 \cup \gamma_3$ of length $\ell (t_1, u_1)$ is a minimizer.

Proof. Again, the result follows from continuity of ℓ. $\square$

We conclude this section with an interesting observation that distinguishes conical cups from soup cans.

Definition 3. *Non-axial points A and B on a surface of revolution are* diaxial *if the angle between Γ_A and Γ_B is π.*

Proposition 8. *A point K on a conical cup is the cone point if and only if K is an endpoint of every minimal path containing it. Furthermore, every minimal path ending at the cone point lies in some plane containing the axis.*

Proof. Suppose that K is an endpoint of every minimal path containing it. But K cannot lie on the side since some ruling would contain it; K cannot lie on the rim since the path $\Gamma_K \cap S$ would contain it; and K cannot lie on the lid since some diameter would contain it. Thus K is the cone point. Conversely, we claim that the cone point K is an endpoint of every minimal path containing it. Suppose that γ is a minimal path from A to B, where $A, B \neq K$. Since minimal paths between two points on the lid or from a point on the side to a point on the lid miss K, we may assume that A and B lie on the side. If A and B are diaxial, let α be a ruling that misses A and B. If not, choose a plane Π containing the axis and bounding an open half-space H with $A, B \in H$ and let α be a ruling in $\Pi \cap S$. In either case, choose a flat model $(\mathfrak{F}, q)$ with q mapping the edges of sector C onto α. If ϕ is the cone angle, $m\angle AKB \leq \pi \sin\phi < \pi$ so that $\overline{AB} \subset C$; and furthermore, $K \notin \overline{AB}$ since $K \subset \partial C$. It follows that $\gamma = q\left(\overline{AB}\right)$ misses K (see Figure 5). Finally, suppose A is the cone point. If B is non-axial, there is a minimal path from A to B in $\Gamma_B \cap S$. If B is axial, choose any point P distinct from A and B; the path $\Gamma_P \cap S$ is minimal. Thus every minimal path beginning at the cone point lies in some plane containing the axis. $\square$
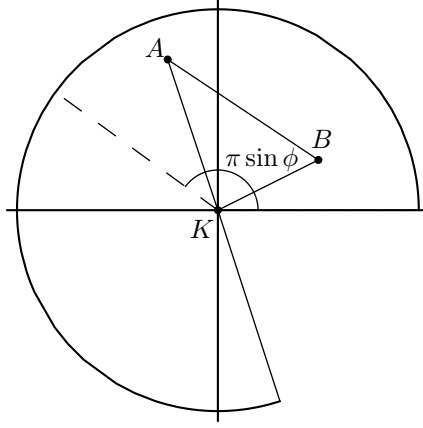

FIGURE 5. A minimal path on the side misses the cone point.

3. NON-UNIQUE MINIMAL PATHS

We conclude the paper with a discussion of some situations in which multiple minimal paths from A to B exist. Consider a soup can S with diaxial points A and B on opposite rims. If height $h < \frac{\pi^2 - 4}{4}$, there are exactly two minimal paths from A to B–one across the base and one across the lid. If $h > \frac{\pi^2 - 4}{4}$, again there are exactly two minimal paths, but this time both are on the side. And if

$h = \frac{\pi^2 - 4}{4}$, there are exactly four–one across the lid, one across the base and two on the side (see Figure 6). In general, given diaxial points A and B, there are at most four minimal paths connecting them; in particular, if height $h > \frac{\pi^2 - 4}{4}$, there are at most three–one across the lid and two around the side. This motivates the following somewhat surprising result for conical cups.
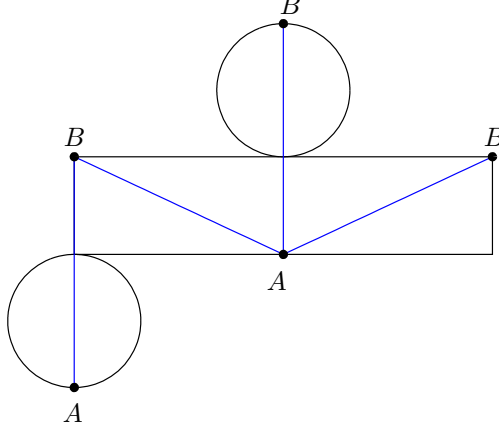

FIGURE 6. Four minimal paths from A to B.

Theorem 3. *The side of a conical cup with slant height s contains diaxial points A and B joined by three minimal paths. Specifically, let $r_{\max}$ be the maximum radius of curvature along an epicycloid generated by a point on the rim as it rolls along the sector arc in a flat model. If A is non-axial point of distance $a \geq s - r_{\max} + 2$ from the cone point, there are three minimal paths from A to the diaxial point B whose distance b from the cone point is*

$$(3.1) \qquad b = \frac{2(s - a + 1)(s + 1)}{2s + 2 - a[1 + \cos(\pi/s)]}.$$

Proof. First observe that minimal paths across the lid from A to B follow a diameter: Consider the flat model in which sector C is centered at the origin with one edge along the positive x-axis, the disk U is centered at $(s + 1, 0)$, $A(s - a, 0)$ and $B((s - b)\cos(\pi/s), (s - b)\sin(\pi/s))$. Let α be the trace of the epicycloid generated by $P(s + 2, 0)$ as U rolls along C. Since the radius of curvature along α attains its maximum $r_{\max}$ at P, the circle centered at A of radius $s - a + 2 \leq r_{\max}$ intersects α only at P and the distance from A to α is minimized at P. Of course, the distance from B to the sector arc is minimized at $Q(s\cos(\pi/s), s\sin(\pi/s))$, so minimal paths across the rim from A to B lie in $(\Gamma_A \cup \Gamma_B) \cap S$ as claimed. To obtain relation (3.1), note that sector C subtends angle $2\pi/s$ and contains $\overline{AB}$. Since A and B are diaxial, $m\angle AOB = \pi/s$ and

$$AB = \sqrt{a^2 - 2ab\cos(\pi/s) + b^2}$$

by the Law of cosines. Now assume for the moment that AB is the minimal length $2s - a - b + 2$ of paths across the lid. Then

$$(2s - a - b + 2)^2 = a^2 - 2ab\cos(\pi/s) + b^2$$

and solving for b gives relation (3.1). So given A such that $a \geq s - r_{\max} + 2$, let b be given by relation (3.1). Choose B at distance b from the cone point and positioned so that A and B are diaxial. Then A and B are joined by exactly three minimal paths (see Figure 7).   □
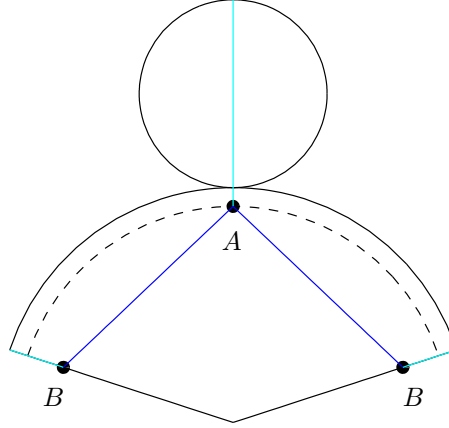

FIGURE 7. Three minimal paths from A to B.

Sufficiently tall soup cans admit an analogous result.

Corollary 2. *Let A and B be diaxial points on the side of a soup can S whose respective distances from the lid are c and d. If S has critical height*

$$h = \frac{\pi^2 + 4c^2 - 4}{4c + 4}$$

and $d = h - c$, there are exactly four minimal paths from A to B. In particular, if A and B lie on opposite rims, $h = \frac{\pi^2 - 4}{4}$; if A and B lie on the mid-circle, $h = \pi - 2$.

Proof. Set $a = s - c$ and $b = s - d$ in relation (3.1) and obtain

$$d = -\frac{2c + 2 + s(c - s)[1 - \cos(\pi/s)]}{2s + 2 + (c - s)[1 + \cos(\pi/s)]}.$$

Since

$$-\lim_{s \to \infty} \frac{2c + 2 + s(c - s)[1 - \cos(\pi/s)]}{2s + 2 + (c - s)[1 + \cos(\pi/s)]} = \frac{\pi^2 - 4c - 4}{4c + 4},$$

the critical height $h = c + \frac{\pi^2 - 4c - 4}{4c + 4}$.   □

Multiple minimal paths can join points in more general position. For example, let P and B be points on the rim of a soup can S such that the angle θ between Γ_P and Γ_B satisfies $0 < \theta_P < \pi/2$. Consider the flat model whose lid is tangent to the rectangular side R at P, and let $B' \in \partial R$ be the point identified with B. For S with sufficient height h, there is a point $A \in \overleftrightarrow{PB} \cap R$ such that $\triangle ABB'$ is isosceles and there are at least two minimal paths in $\Omega_{A,B}$ from A to B–one across the lid and one around the side. In fact, there is a critical height h at which there are three minimal paths–the two just mentioned and a third across the base. Computing this critical height h requires more machinery than we needed above, but is similar in

spirit. We conclude with a computation of h when A and B lie on the mid-circle of a soup can. We need the following well-known fact about roulettes:

Proposition 9. *Given a point B and curves C_1 and C_2 as in Definition 2, let β be the roulette generated by B. If $\|\beta'(t)\| > 0$, the line normal to β at $\beta(t)$ passes through the point of tangency $\alpha_2(t)$.*

Now if A and B are joined by a minimal path γ_1 crossing the rim T_1 at P and Q, the mirror image of γ_1 is a minimal path γ_2 across the base U_2. We wish to compute the critical height at which there is a third path γ_3 of the same length around the side. Assume that Q is the point on T_1 closest to B, and let β be the cycloid generated by Q as U_1 rolls along its edge of rectangle R in a flat model. Let n be the line normal to β passing through A and let θ be its angle of inclination. When T_1 is tangent to R at P, the central angle in T_1 subtended by the chord of n is 2θ and the height of the soup can is $h = 2BP\sin\theta$. So if h is the desired critical height, $\theta = \sin^{-1}\left(\frac{h}{2BP}\right)$ is a solution of

$$(3.2) \qquad\qquad \theta - \sin\theta = BP\left(1 - \cos\theta\right).$$

Conversely, solving equation (3.2) for θ determines h. Other cases are similar in spirit and left to the reader.

References

[1] T. Banchoff. *Minimal Submanifolds of the Bicylinder Boundary.* Boletim da Sociedade, Brasileira de Matematica (1976), 37-57.
[2] A. Cotton, D. Freeman, A. Gnepp, T. Ng, J. Spivack, C. Yoder. *The Isoperimetric Problem on Some Singular Surfaces.* J. Australian Math. Soc. 78 (2005), 167-197.
[3] J. Mohler. *Distance Minimizing Paths on Certain Simple Flat Surfaces with Singularities,* Undergraduate Honors Thesis, Millersville University of PA, 2002.
[4] J. Munkres. "Topology", 2nd Ed. Prentice Hall, Upper Saddle River, NJ, 2000.
[5] B. O'Neill. "Elementary Differential Geometry", Academic Press, New York, 1966.

Department of Mathematics, Lehigh University, Bethlehem, PA 18015
E-mail address: `jbm5@lehigh.edu`
URL: `http://www.lehigh.edu/~jbm5/`

Department of Mathematics, Millersville University, Millersville, PA 17551
E-mail address: `ron.umble@millersville.edu`
URL: `http://www.millersville.edu/~rumble`